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2002 J. Phys. A: Math. Gen. 35 3343

(http://iopscience.iop.org/0305-4470/35/14/314)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 3343–3356 PII: S0305-4470(02)33091-9

Bose realization of the $Sp(4) \supset SU(2) \times U(1)$ tensor operators

R A Tello-Llanos

Departamento de Formación General y Ciencias Básicas, Universidad Simón Bolívar, Apdo 89000, Caracas 1080A, Venezuela

E-mail: rtello@usb.ve

Received 23 January 2001
Published 29 March 2002
Online at stacks.iop.org/JPhysA/35/3343

Abstract
A realization of irreducible tensor operators under the non-canonical reduction of the compact symplectic group $Sp(4) \sim O(5) \supset SU(2) \times U(1)$ is developed in terms of a system of Bose creation and annihilation operators. The method of derivation of their matrix elements is presented and some concrete cases are explicitly computed as illustration.

PACS numbers: 02.20.Qs, 02.20.−a, 02.20.Uw

1. Introduction

Irreducible tensor operators are defined as a set of operators transforming among themselves according to an irreducible representation of a given group. In the case of the non-canonical reduction of the compact symplectic group $Sp(4) \sim O(5) \supset SU(2) \times U(1)$, the lack of one parameter for the complete specification of the basis vectors, within an arbitrary irreducible representation, makes obtaining the matrices of generic representations and the very definition of tensor operators difficult. However, the operators that appear in physical applications transform according to irreducible representations of low dimension, for which the missing label problem is absent and the tensor operators are well defined. In the present article we will deal exclusively with this kind of operator.

The generators of $Sp(4)$ can be denoted as G^a_b, $a, b = -2, \ldots, 2$, zero excluded [1]. They satisfy the identities $G^a_b = -\epsilon^a \epsilon^b G^{-b}_{-a}$, with $\epsilon^a \equiv a/|a|$. In unitary representations $G^{a+}_b = G^b_a$. Under the reduction $Sp(4) \supset SU(2) \times U(1)$ the generators of the group can be defined as follows (for the convenience of the reader, the relation with the $O(5)$ Hermitian generators $L_{jk} = -\mathrm{i}(x_j \nabla_k - x_k \nabla_j)$, $j, k = 1, \ldots, 5$, is also given): the $SU(2)$ generators are $J_0 = \left(G^1_1 - G^2_2\right)/2 = L_{34}$, $J_+ = G^1_2 = L_{45} + \mathrm{i}L_{53}$ and $J_- = G^2_1 = L_{45} - \mathrm{i}L_{53}$. The $U(1)$ generator is $H = \left(G^1_1 + G^2_2\right)/2 = L_{12}$. The remaining generators define two irreducible vector operators with respect to the $SU(2)$ subgroup, one of which has the tensor components $U_1 = G^1_{-1}/\sqrt{2} = [L_{14} + L_{23} + \mathrm{i}(L_{24} + L_{31})]/\sqrt{2}$, $U_0 = G^1_{-2} = L_{52} + \mathrm{i}L_{15}$,

$U_{-1} = G^2_{-2}/\sqrt{2} = [L_{14} - L_{23} + \mathrm{i}\,(L_{24} - L_{31})]/\sqrt{2}$; the other, Hermitian conjugated, has the components $V_\kappa = (-1)^\kappa U^+_{-\kappa}$, $\kappa = 0, \pm 1$. The irreducible unitary representations can be labelled by (ω_1, ω_2), where ω_1 and ω_2 are the eigenvalues of the operators H and J_0 in the highest weight state. There are three available parameters for the identification of the basis states: the eigenvalues of the commuting Hermitian operators $\mathbf{J}^2$, J_0 and H, which will be denoted as $j\,(j+1)$, m and τ. The necessary fourth parameter is missing; and this fact has been the essential problem in the construction of the non-canonical representations of $Sp(4) \supset SU(2) \times U(1)$.

Following Hecht [2], the irreducible tensor operators $T^{(\omega_1,\omega_2)}_{\tau jm}$ are defined through their commutation relations with the generators of the group

$$\left[H, T^{(\omega_1,\omega_2)}_{\tau jm}\right] = \tau T^{(\omega_1,\omega_2)}_{\tau jm} \tag{1}$$

$$\left[J_0, T^{(\omega_1,\omega_2)}_{\tau jm}\right] = m T^{(\omega_1,\omega_2)}_{\tau jm} \tag{2}$$

$$\left[J_\pm, T^{(\omega_1,\omega_2)}_{\tau jm}\right] = \sqrt{j(j+1) - m(m\pm 1)}\,T^{(\omega_1,\omega_2)}_{\tau jm\pm 1} \tag{3}$$

$$\left[U_\kappa, T^{(\omega_1,\omega_2)}_{\tau jm}\right] = \sum_{j'} T^{(\omega_1,\omega_2)}_{\tau+1,j',m+\kappa} \langle(\omega_1,\omega_2)\tau+1, j', m+\kappa|U_\kappa|(\omega_1,\omega_2)\tau, j, m\rangle \tag{4}$$

$$\left[V_\kappa, T^{(\omega_1,\omega_2)}_{\tau jm}\right] = \sum_{j'} T^{(\omega_1,\omega_2)}_{\tau-1,j',m+\kappa} \langle(\omega_1,\omega_2)\tau-1, j', m+\kappa|V_\kappa|(\omega_1,\omega_2)\tau, j, m\rangle. \tag{5}$$

Note that in equations (4) and (5) the basis states of representation (ω_1, ω_2) are labelled with only three parameters τ, j, m without a multiplicity label. This will be sufficient for the representations $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(1, 0)$, $(1, 1)$, $(2, 0)$ and $(2, 2)$, of remarkable interest in physical applications, as was discussed in detail by Hecht [2].

In the present paper a new and complete description of the irreducible tensor operators which transform according to $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(1, 0)$ and $(1, 1)$ will be given as well as a method for the derivation of their matrix elements. The tensor operators transforming as $(2, 0)$ and $(2, 2)$ could be treated with the same method, but were left out of these investigation for brevity.

The necessary computations are based on a Bose realization of the representation space and the algebra of generators recently proposed by the present author [3], as well as on a straightforward use of the $SU(2)$ tensor algebra.

2. Bose realization of $Sp(4) \supset SU(2) \times U(1)$

In [3] a Bose realization of the non-canonical representations of $Sp(4) \supset SU(2) \times U(1)$ was obtained. The generators of the algebra were expressed as linear combinations of the Weyl generators E_{ij} of $SU(4)$ $(i, j = 1, \ldots, 4)$:

$$H = (E_{11} - E_{22} + E_{33} - E_{44})/2 \tag{6}$$

$$J_0 = (E_{11} - E_{22} - E_{33} + E_{44})/2 \qquad J_+ = E_{13} - E_{42} \qquad J_- = E_{31} - E_{24} \tag{7}$$

$$U_0 = E_{14} + E_{32} \qquad U_1 = \sqrt{2}E_{12} \qquad U_{-1} = \sqrt{2}E_{34} \tag{8}$$

$$V_0 = E_{41} + E_{23} \qquad V_1 = -\sqrt{2}E_{43} \qquad V_{-1} = -\sqrt{2}E_{21}. \tag{9}$$

Following a prior proposal due to Holman [4], the Bose realization was defined through a Schwinger-type mapping of E_{ij} onto the union of a set of boson creation $a^1_i(1)$ and destruction $\bar{a}^1_i(1)$ operators with another independent set $a^b_i(2)$ and $\bar{a}^b_i(2)$ $(i = 1, \ldots, 4$ and $b = 1, 2)$:

$$E_{ij} = a^1_i(1)\bar{a}^1_j(1) + a^1_i(2)\bar{a}^1_j(2) + a^2_i(2)\bar{a}^2_j(2). \tag{10}$$

In this realization, the basis of a generic ($\omega_1 = q + p/2, \omega_2 = p/2$) representation is a subspace of the product of representation spaces of $(p/2, p/2)$ and $(q, 0)$, according to the rule

$$|(q + p/2, p/2)\alpha n JM\rangle = \sum_{\{n_1 n_2 j_1 j_2\}} |pqn_1n_2 j_1 j_2 nJM\rangle A_{\alpha nJ}(n_1n_2 j_1 j_2). \quad (11)$$

For convenience, the notation $\tau \equiv n,\ \tau_1 \equiv n_1,\ \tau_2 \equiv 2n_2$ was introduced, so that $n = n_1 + 2n_2$. The parameter α enumerates the degeneracies of the basis vectors. The vectors on the right-hand side form a standard basis of $(p/2, p/2) \times (q, 0)$:

$$|pqn_1n_2 j_1 j_2 nJM\rangle = \sum_{m_1m_2} \langle j_1 j_2 m_1 m_2\ |JM\rangle |(p/2, p/2)n_1 j_1 m_1\rangle |(q, 0)n_2 j_2 m_2\rangle. \quad (12)$$

The basis vectors in $(p/2, p/2)$ and $(q, 0)$ are defined as polynomials in two independent sets of boson operators, applied to the boson vacuum: the set of single creation operators $a^1_1(1), a^1_2(1), a^1_3(1), a^1_4(1)$, and the set of double antisymmetric creation operators $a_{14}(2), a_{21}(2) + a_{34}(2), a_{23}(2), a_{13}(2), a_{24}(2)$, respectively. For details, see equations (12) and (20) of [3]. The double boson operators are defined as

$$a_{ij}(2) = a^1_i(2)a^2_j(2) - a^1_j(2)a^2_i(2) \quad (13)$$

and, similarly, the double destruction operators $\bar{a}_{ij}(2) = [a_{ij}(2)]^+$.

The coefficients $A_{\alpha nJ}(n_1n_2 j_1 j_2)$ are the solution of a system of linear homogeneous algebraic equations, derived in terms of a set of elementary tensor operators under the $SU(2)$ subgroup of $Sp(4)$. These are denoted by $\mathbf{Q}^{(1/2)}$, $\mathbf{R}^{(1/2)}$, $\mathbf{S}^{(1)}$, and the scalars $\bar{a}_{13}$, $\bar{a}_{24}$. These operators are used below with a choice of their phase factors slightly different from the one made in [3]. Their components are defined as follows, maintaining, for brevity, their rank in implicit form:

$$Q_{1/2} = -\mathrm{i}\bar{a}^1_3(1) \qquad Q_{-1/2} = \mathrm{i}\bar{a}^1_1(1) \qquad R_{1/2} = \bar{a}^1_2(1) \qquad R_{-1/2} = \bar{a}^1_4(1) \quad (14)$$

$$S_1 = -\bar{a}_{23} \qquad S_0 = (\bar{a}_{21} + \bar{a}_{34})/\sqrt{2} \qquad S_{-1} = -\bar{a}_{14}. \quad (15)$$

In the next section, these destruction operators, as well as the Hermitian-conjugated creation operators, are combined in $SU(2)$ tensor products in order to describe the solutions of the equations (1)–(5). The Hermitian conjugation is defined as usual: $\mathbf{B}^{(k)} = \mathbf{T}^{(k)+}$ if $B^{(k)}_q = (-1)^q \big(T^{(k)}_{-q}\big)^+$. For example, the Hermitian-conjugated creation operators $\mathbf{Q}^{(1/2)+}$, $\mathbf{R}^{(1/2)+}$ and $\mathbf{S}^{(1)+}$ have the components

$$Q^+_{1/2} = a^1_1(1) \qquad Q^+_{-1/2} = a^1_3(1) \qquad R^+_{1/2} = \mathrm{i}a^1_4(1) \qquad R^+_{-1/2} = -\mathrm{i}a^1_2(1) \quad (16)$$

$$S^+_1 = a_{14} \qquad S^+_0 = (a_{21} + a_{34})/\sqrt{2} \qquad S^+_{-1} = a_{23}. \quad (17)$$

Their reduced matrix elements, although given in [3], are presented in appendix B with the new definition of phase factors.

3. Irreducible shift tensor operators

As was stressed by Hecht [5], $Sp(4) \sim SO(5)$ irreducible tensor operators are classified not only by their rank (ω_1, ω_2), but also by the shift (Δ_1, Δ_2) they induce in an irreducible representation, when acting on a generic state

$$T^{(\omega_1\omega_2)}_{\tau jm}(\Delta_1, \Delta_2)|(\omega'_1\omega'_2)\alpha n JM\rangle \to |(\omega'_1 + \Delta_1, \omega'_2 + \Delta_2)\alpha' n + \tau J'M + m\rangle. \quad (18)$$

The parameters Δ_1 and Δ_2 take the same set of values that τ and m take within a given $Sp(4)$ irreducible representation $(\omega_1\omega_2)$.

Table 1. Irreducible shift tensors $T^{(\frac{1}{2},\frac{1}{2})}_{njm}(\Delta_1,\Delta_2)$.

Δ_1	Δ_2	$T^{(\frac{1}{2},\frac{1}{2})}_{\frac{1}{2}\frac{1}{2}m}(\Delta_1,\Delta_2)$	$T^{(\frac{1}{2},\frac{1}{2})}_{-\frac{1}{2}\frac{1}{2}m}(\Delta_1,\Delta_2)$
$\frac{1}{2}$	$\frac{1}{2}$	Q^+_m	R^+_m
$\frac{1}{2}$	$-\frac{1}{2}$	$a_{13}Q_m - \mathrm{i}\sqrt{\frac{3}{2}}[S^+\times R]^{(1/2)}_m$	$-a_{24}R_m + \mathrm{i}\sqrt{\frac{3}{2}}[S^+\times Q]^{(1/2)}_m$
$-\frac{1}{2}$	$\frac{1}{2}$	$-R^+_m\bar{a}_{24} - \mathrm{i}\sqrt{\frac{3}{2}}[Q^+\times S]^{(1/2)}_m$	$Q^+_m\bar{a}_{13} + \mathrm{i}\sqrt{\frac{3}{2}}[R^+\times S]^{(1/2)}_m$
$-\frac{1}{2}$	$-\frac{1}{2}$	R_m	Q_m

The correspondence (18) with $\tau = \Delta_1$ and $m = \Delta_2$, applied to the highest weight

$$|(\omega'_1\omega'_2)n = \omega'_1, J = M = \omega'_2\rangle = \frac{\left[a^1_1(1)\right]^{2\omega'_2}[a_{13}(2)]^{\omega'_1-\omega'_2}}{[(2\omega'_2)!(\omega'_1-\omega'_2)!(\omega'_1-\omega'_2+1)!]^{1/2}}|0\rangle \tag{19}$$

shows that $T^{(\omega_1\omega_2)}_{\Delta_1 j\Delta_2}(\Delta_1,\Delta_2)$ behaves as $\left[a^1_1(1)\right]^{2\Delta_2}[a_{13}(2)]^{\Delta_1-\Delta_2}$ on this state. The derivation of these operators can be completed applying the commutation relations (1)–(5). The results are shown in the next sections in terms of the elementary tensors of the previous section and the usual definition of the $SU(2)$ tensor product

$$[\mathbf{U}\times\mathbf{V}]^{(K)}_Q = \sum_{q_1q_2}\langle k_1k_2q_1q_2|KQ\rangle\, U^{(k_1)}_{q_1}V^{(k_2)}_{q_2}. \tag{20}$$

3.1. Tensors $T^{(\frac{1}{2},\frac{1}{2})}_{njm}(\Delta_1,\Delta_2)$

The components of $\left(\frac{1}{2},\frac{1}{2}\right)$ tensors are shown in table 1. Their parameters take the values $j = \frac{1}{2}$ and $n, m = \pm 1/2$.

3.2. Tensors $T^{(1,0)}_{njm}(\Delta_1,\Delta_2)$

In the case of $(1,0)$ tensors the parameters take the values: $n = 1$, $m = 0$ for $j = 0$ and $n = 0$, $m = 1, 0, -1$ for $j = 1$. Their components are given by the expressions

$$T^{(1,0)}_{100}(1,0) = a_{13} \qquad T^{(1,0)}_{01m}(1,0) = S^+_m \qquad T^{(1,0)}_{-100}(1,0) = a_{24} \tag{21}$$

$$\begin{aligned}
T^{(1,0)}_{100}(0,1) &= -\mathrm{i}\sqrt{2}[Q^+\times R^+]^{(0)}\bar{a}_{24} + \sqrt{3}\left[[Q^+\times Q^+]^{(1)}\times S\right]^{(0)}\\
T^{(1,0)}_{01m}(0,1) &= [Q^+\times Q^+]^{(1)}_m\bar{a}_{13} + [R^+\times R^+]^{(1)}_m\bar{a}_{24} + 2\mathrm{i}\left[[Q^+\times R^+]^{(1)}\times S\right]^{(1)}_m\\
T^{(1,0)}_{-100}(0,1) &= \mathrm{i}\sqrt{2}[Q^+\times R^+]^{(0)}\bar{a}_{13} + \sqrt{3}\left[[R^+\times R^+]^{(1)}\times S\right]^{(0)}
\end{aligned} \tag{22}$$

$$\begin{aligned}
T^{(1,0)}_{100}(0,0) &= \sqrt{2}[Q^+\times R]^{(0)}\\
T^{(1,0)}_{01m}(0,0) &= \mathrm{i}[R^+\times R]^{(1)}_m - \mathrm{i}[Q^+\times Q]^{(1)}_m\\
T^{(1,0)}_{-100}(0,0) &= -\sqrt{2}[R^+\times Q]^{(0)}.
\end{aligned} \tag{23}$$

The remaining tensors $T^{(1,0)}_{njm}(0,-1)$ and $T^{(1,0)}_{njm}(-1,0)$ can be derived from equations (22) and (21) by Hermitian conjugation, using the property

$$T^{(\omega_1,\omega_2)}_{njm}(\Delta_1,\Delta_2) = T^{(\omega_1,\omega_2)}_{-njm}(-\Delta_1,-\Delta_2)^+. \tag{24}$$

3.3. Tensors $T^{(1,1)}_{njm}(\Delta_1, \Delta_2)$

The parameters of (1, 1) tensors take the values $n, m = 1, 0, -1$ for $j = 1$ and $n, m = 0$ for $j = 0$. Their components are obtained in the form

$$\begin{aligned}
T^{(1,1)}_{11m}(1,1) &= [Q^+ \times Q^+]^{(1)}_m \\
T^{(1,1)}_{01m}(1,1) &= -\mathrm{i}[Q^+ \times R^+]^{(1)}_m \\
T^{(1,1)}_{-11m}(1,1) &= -[R^+ \times R^+]^{(1)}_m \\
T^{(1,1)}_{000}(1,1) &= -\mathrm{i}[Q^+ \times R^+]^{(0)}
\end{aligned} \tag{25}$$

$$\begin{aligned}
T^{(1,1)}_{11m}(1,0) &= \mathrm{i}a_{13}\left\{[R^+ \times R]^{(1)}_m - [Q^+ \times Q]^{(1)}_m\right\} - \sqrt{2}S^+_m[Q^+ \times R]^{(0)} \\
T^{(1,1)}_{01m}(1,0) &= \mathrm{i}\left[S^+ \times [R^+ \times R]^{(1)}\right]^{(1)}_m - \mathrm{i}\left[S^+ \times [Q^+ \times Q]^{(1)}\right]^{(1)}_m \\
T^{(1,1)}_{-11m}(1,0) &= \mathrm{i}a_{24}\left\{[Q^+ \times Q]^{(1)}_m - [R^+ \times R]^{(1)}_m\right\} - \sqrt{2}S^+_m[R^+ \times Q]^{(0)} \\
T^{(1,1)}_{000}(1,0) &= a_{13}[R^+ \times Q]^{(0)} + a_{24}[Q^+ \times R]^{(0)}
\end{aligned} \tag{26}$$

$$\begin{aligned}
T^{(1,1)}_{11m}(1,-1) &= -a^2_{13}[Q \times Q]^{(1)}_m + 2\mathrm{i}a_{13}\left[S^+ \times [R \times Q]^{(1)}\right]^{(1)}_m - \mathrm{i}\sqrt{2}a_{13}S^+_m[R \times Q]^{(0)} \\
&\quad + \sqrt{3}S^+_m\left[S^+ \times [R \times R]^{(1)}\right]^{(0)} - \frac{\sqrt{3}}{2}[S^+ \times S^+]^{(0)}[R \times R]^{(1)}_m \\
T^{(1,1)}_{01m}(1,-1) &= -a_{13}\left[S^+ \times [Q \times Q]^{(1)}\right]^{(1)}_m - a_{24}\left[S^+ \times [R \times R]^{(1)}\right]^{(1)}_m - \mathrm{i}a_{13}a_{24}[R \times Q]^{(1)}_m \\
&\quad + \mathrm{i}\sqrt{3}S^+_m\left[S^+ \times [R \times Q]^{(1)}\right]^{(0)} - \mathrm{i}\frac{\sqrt{3}}{2}[S^+ \times S^+]^{(0)}[R \times Q]^{(1)}_m \\
T^{(1,1)}_{-11m}(1,-1) &= a^2_{24}[R \times R]^{(1)}_m - 2\mathrm{i}a_{24}\left[S^+ \times [R \times Q]^{(1)}\right]^{(1)}_m - \mathrm{i}\sqrt{2}a_{24}S^+_m[R \times Q]^{(0)} \\
&\quad - \sqrt{3}S^+_m\left[S^+ \times [Q \times Q]^{(1)}\right]^{(0)} + \frac{\sqrt{3}}{2}[S^+ \times S^+]^{(0)}[Q \times Q]^{(1)}_m \\
T^{(1,1)}_{000}(1,-1) &= \mathrm{i}a_{13}a_{24}[R \times Q]^{(0)} + \sqrt{\frac{3}{2}}a_{13}\left[S^+ \times [Q \times Q]^{(1)}\right]^{(0)} \\
&\quad - \sqrt{\frac{3}{2}}a_{24}\left[S^+ \times [R \times R]^{(1)}\right]^{(0)} + \frac{\sqrt{3}}{2}\mathrm{i}[S^+ \times S^+]^{(0)}[R \times Q]^{(0)}
\end{aligned} \tag{27}$$

$$\begin{aligned}
T^{(1,1)}_{11m}(0,1) &= 2\left[[Q^+ \times Q^+]^{(1)} \times S\right]^{(1)}_m - 2\mathrm{i}[Q^+ \times R^+]^{(1)}_m\bar{a}_{24} \\
T^{(1,1)}_{01m}(0,1) &= [Q^+ \times Q^+]^{(1)}_m\bar{a}_{13} - [R^+ \times R^+]^{(1)}_m\bar{a}_{24} - \mathrm{i}\sqrt{2}[Q^+ \times R^+]^{(0)}S_m \\
T^{(1,1)}_{-11m}(0,1) &= 2\left[[R^+ \times R^+]^{(1)} \times S\right]^{(1)}_m - 2\mathrm{i}[Q^+ \times R^+]^{(1)}_m\bar{a}_{13} \\
T^{(1,1)}_{000}(0,1) &= \mathrm{i}\sqrt{6}\left[[Q^+ \times R^+]^{(1)} \times S\right]^{(0)}.
\end{aligned} \tag{28}$$

There are two independent solutions for the $(\Delta_1 = 0, \Delta_2 = 0)$ shift, one defined on the representation space of $(p/2, p/2)$ and the other on $(q, 0)$:

$$\begin{aligned}
T^{(1,1)}_{11m}(0,0)_1 &= 2\sqrt{2}\mathrm{i}[Q^+ \times R]^{(1)}_m \\
T^{(1,1)}_{01m}(0,0)_1 &= \sqrt{2}[Q^+ \times Q]^{(1)}_m + \sqrt{2}[R^+ \times R]^{(1)}_m \\
T^{(1,1)}_{-11m}(0,0)_1 &= -2\sqrt{2}\mathrm{i}[R^+ \times Q]^{(1)}_m \\
T^{(1,1)}_{000}(0,0)_1 &= \sqrt{2}[Q^+ \times Q]^{(0)} - \sqrt{2}[R^+ \times R]^{(0)}
\end{aligned} \tag{29}$$

$$\begin{aligned}
T^{(1,1)}_{11m}(0,0)_2 &= -\sqrt{2}[a_{13}, S_m] + \sqrt{2}[S^+_m, \bar{a}_{24}]\\
T^{(1,1)}_{01m}(0,0)_2 &= -\sqrt{2}[S^+ \times S]^{(1)}_m - \sqrt{2}[S \times S^+]^{(1)}_m\\
T^{(1,1)}_{-11m}(0,0)_2 &= \sqrt{2}[a_{24}, S_m] - \sqrt{2}[S^+_m, \bar{a}_{13}]\\
T^{(1,1)}_{000}(0,0)_2 &= [a_{13}, \bar{a}_{13}] - [a_{24}, \bar{a}_{24}].
\end{aligned} \tag{30}$$

Note that in (30) the commutator of two operators was applied. The remaining tensors $T^{(1,1)}_{njm}(\Delta_1, \Delta_2)$, with $(\Delta_1, \Delta_2) = (0, -1), (-1, 1), (-1, 0)$ and $(-1, -1)$ can be obtained by Hermitian conjugation of equations (28), (27), (26) and (25), using the property (24).

Finally, the generators (6)–(9) of $Sp(4)$ themselves can be expressed in terms of the $(1, 1)$ tensors of this section:

$$H = -\frac{\mathrm{i}}{2}T^{(1,1)}_{000}(0,0)_1 - \frac{1}{2}T^{(1,1)}_{000}(0,0)_2 \tag{31}$$

$$J_m = -\frac{\mathrm{i}}{2}T^{(1,1)}_{01m}(0,0)_1 - \frac{1}{2}T^{(1,1)}_{01m}(0,0)_2 \tag{32}$$

$$U_m = -\frac{\mathrm{i}}{2}T^{(1,1)}_{11m}(0,0)_1 - \frac{1}{2}T^{(1,1)}_{11m}(0,0)_2 \tag{33}$$

$$V_m = -\frac{\mathrm{i}}{2}T^{(1,1)}_{-11m}(0,0)_1 - \frac{1}{2}T^{(1,1)}_{-11m}(0,0)_2. \tag{34}$$

4. Matrix elements of tensor operators

Matrix elements of the tensors of the previous section are of great interest; properly normalized they give the $Sp(4) \sim SO(5)$ Wigner coefficients [5]. As can be seen, all of them have a well-defined tensor rank, not only under the $Sp(4)$ group, but also under the $SU(2)$ subgroup. Thanks to the $SU(2)$ Wigner–Eckart theorem, their matrix elements take the form

$$\begin{aligned}
&\left\langle(\omega''_1\omega''_2)\alpha''n''J''M''\middle|T^{(\omega_1\omega_2)}_{njm}\middle|(\omega'_1\omega'_2)\alpha'n'J'M'\right\rangle\\
&\quad = (-1)^{J''-M''}\begin{pmatrix} J'' & j & J'\\ -M'' & m & M'\end{pmatrix}\left\langle(\omega''_1\omega''_2)\alpha''n''J''\middle\|\mathbf{T}^{(\omega_1\omega_2)}_{nj}\middle\|(\omega'_1\omega'_2)\alpha'n'J'\right\rangle.
\end{aligned} \tag{35}$$

In this equation the reduced matrix elements are given by

$$\begin{aligned}
&\left\langle(\omega''_1\omega''_2)\alpha''n''J''\middle\|\mathbf{T}^{(\omega_1\omega_2)}_{nj}\middle\|(\omega'_1\omega'_2)\alpha'n'J'\right\rangle = \sum\sum A''^{*}_{\alpha''n''J''}(n''_1n''_2j''_1j''_2)\\
&\quad\times\left\langle p''q''n''_1n''_2j''_1j''_2J''\middle\|\mathbf{T}^{(\omega_1\omega_2)}_{nj}\middle\|p'q'n'_1n'_2j'_1j'_2J'\right\rangle A'_{\alpha'n'J'}(n'_1n'_2j'_1j'_2).
\end{aligned} \tag{36}$$

Here, $p'' = 2\omega''_2$, $q'' = \omega''_1 - \omega''_2$ and, similarly, $p' = 2\omega'_2$, $q' = \omega'_1 - \omega'_2$. The sums are fulfilled over the sets $\{n''_1n''_2j''_1j''_2\}$ and $\{n'_1n'_2j'_1j'_2\}$ that lead to the same values $n''J''$ and $n'J'$ in the products of representations $(p''/2, p''/2) \times (q'', 0)$ and $(p'/2, p'/2) \times (q', 0)$, as was described in [3]. The matrix elements on the right-hand side of (36) are defined in the standard bases (12) of the product of representations . They can be computed with the usual composition formulae of the $SU(2)$ tensor algebra, as will be shown in the following examples. The coefficients $A_{\alpha nJ}$ are responsible for the projection $(p/2, p/2) \times (q, 0) \rightarrow (q + p/2, p/2)$. These coefficients are also exemplified below.

4.1. Examples

The first example deals with one of the simplest cases:

$$T^{(\frac{1}{2}\frac{1}{2})}_{\tau jm}(1/2, 1/2)|(q+1, 1)\alpha nJM\rangle \to |(q+3/2, 3/2)\alpha', n+\tau, J', M+m\rangle. \tag{37}$$

From table 1, this shift tensor has the components Q^{+}_{m} and R^{+}_{m}. Let us consider the matrix elements of $\mathbf{Q}^{+}$ ($\mathbf{R}^{+}$ can be treated in complete analogy). Its reduced matrix elements are defined as

$$\begin{aligned}\langle (q+3/2, 3/2)\alpha' n' J' \| \mathbf{Q}^{+} \| (q+1, 1)\alpha n J\rangle &= \sum A'^{\star}_{\alpha' n' J'}(n'_1 n'_2 j'_1 j'_2)\\ &\times \langle 3q n'_1 n'_2 j'_1 j'_2 J' \| \mathbf{Q}^{+} \| 2q n_1 n_2 j_1 j_2 J\rangle A_{\alpha n J}(n_1 n_2 j_1 j_2).\end{aligned} \tag{38}$$

The action of $\mathbf{Q}^{+}$ is restricted to the representations $(p/2, p/2)$, $p = 2, 3$, so that the matrix element on the right-hand side can be computed with the use of a *6J* symbol and the reduced matrix elements given in appendix B:

$$\begin{aligned}\langle 3q n'_1 n'_2 j'_1 j'_2 J' \| \mathbf{Q}^{+} \| 2q n_1 n_2 j_1 j_2 J\rangle &= \delta_{n'_2 n_2}\delta_{j'_2 j_2}(-1)^{J+j'_1+j'_2+1/2}\\ &\times \sqrt{(2J'+1)(2J+1)} \begin{Bmatrix} j'_1 & \frac{1}{2} & j_1 \\ J & j'_2 & J' \end{Bmatrix} \langle (3/2, 3/2) n'_1 j'_1 \| \mathbf{Q}^{+} \| (1, 1) n_1 j_1\rangle.\end{aligned} \tag{39}$$

The coefficients $A_{\alpha nJ}(n_1 n_2 j_1 j_2)$ in (38) are solutions of the algebraic equations of [3]. In general, the cases with $q+J+n = odd$ and $q+J+n = even$ have to be distinguished. These are shown in appendix A: tables A2 and A3 for the representations $(q+1, 1)$ and tables A4 and A5 for $(q+3/2, 3/2)$.

The final expressions for the non-zero reduced matrix elements of $\mathbf{Q}^{+}$ and $\mathbf{R}^{+}$ are given in tables 2 and 3.

As a second example, the following shift can be considered:

$$T^{(\frac{1}{2},\frac{1}{2})}_{\tau jm}(1/2, -1/2)|(q+1, 1)\alpha nJM\rangle \to |(q'+1/2, 1/2)n+\tau, J', M+m\rangle. \tag{40}$$

Here, and in equations (41)–(43), $q' \equiv q+1$. The components of this shift tensor were defined above in table 1. It is sufficient to show the computation of the matrix elements of the operators $T^{(\frac{1}{2},\frac{1}{2})}_{1/2,1/2,m}(1/2, -1/2)$, $m = \pm 1/2$. The other components can be dealt with analogously. According to equation (36), its reduced matrix elements are

$$\begin{aligned}\left\langle (q'+1/2, 1/2)n'J' \middle\| \mathbf{T}^{(\frac{1}{2},\frac{1}{2})}_{1/2,1/2}(1/2, -1/2) \middle\| (q+1, 1)\alpha nJ\right\rangle &= \sum A'^{\star}_{n'J'}(n'_1 n'_2 j'_1 j'_2)\\ &\times \left\langle 1q' n'_1 n'_2 j'_1 j'_2 J' \middle\| \mathbf{Q}a_{13} + \mathrm{i}\sqrt{\tfrac{3}{2}}[\mathbf{R}\times\mathbf{S}^{+}]^{(\frac{1}{2})} \middle\| 2q n_1 n_2 j_1 j_2 J\right\rangle A_{\alpha nJ}(n_1 n_2 j_1 j_2).\end{aligned} \tag{41}$$

Note that the basis of $(q+1/2, 1/2)$ is not degenerate, so that the label α' is irrelevant.

The reduced matrix elements on the right-hand side of equation (41) can be computed with the usual $SU(2)$ composition formulae

$$\begin{aligned}\langle 1q' n'_1 n'_2 j'_1 j'_2 J' \| \mathbf{Q}a_{13} \| 2q n_1 n_2 j_1 j_2 J\rangle &= \delta_{j'_2 j_2}(-1)^{J+j'_1+j_2+1/2}\sqrt{\frac{(2J'+1)(2J+1)}{2j_2+1}}\\ &\times \begin{Bmatrix} j'_1 & j_1 & \frac{1}{2} \\ J & J' & j_2 \end{Bmatrix} \langle (1/2, 1/2) n'_1 j'_1 \| \mathbf{Q} \| (1, 1) n_1 j_1\rangle \langle (q', 0) n'_2 j'_2 \| a_{13} \| (q, 0) n_2 j_2\rangle\end{aligned} \tag{42}$$

Table 2. Matrix elements $\langle (q+3/2, 3/2)\alpha', n+1/2, J+k \| \mathbf{Q}^{+} \| (q+1, 1)\alpha n J\rangle^{a,b}$.

α'	k	α	$q+n+J=$ even
1	$\frac{1}{2}$	1	$\left[\frac{12(J+1)(J+2)(q-J+n+2)(q+J+n+5)[(q+2)(q+3)+J(q+2)-n(q+3)]^2}{(2q+5)G(q,-n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
1	$\frac{1}{2}$	2	$\left[\frac{12J(J+2)(q+2)(q+3)(q-J-n+2)(q+J-n+3)(q+J+n+3)(q+J+n+5)}{(2q+5)G(q,-n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
2	$\frac{1}{2}$	1	$-\left[\frac{4J(J+1)(q+2)(q-J-n+2)(q+J-n+3)[q^2+8q+18+J(q+5)+n(q+3)]^2}{(q+4)(2q+5)G(q,-n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
2	$\frac{1}{2}$	2	$\left[\frac{4(q+3)(q-J+n+2)(q+J+n+3)[3(q+2)(q+4)+J^2(q+2)+J(q^2+8q+12)-Jn(q+5)-3n(q+4)]^2}{(q+4)(2q+5)G(q,-n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
1	$-\frac{1}{2}$	1	$\mathrm{i}\left[\frac{12J(J+1)(q-J-n+2)(q-J+n+4)[(q+2)(q+3)+J(q+2)+n(q+3)]^2}{(2q+5)G(q,n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
1	$-\frac{1}{2}$	2	$-\mathrm{i}\left[\frac{12J^2(q+2)(q+3)(q-J+n+2)(q-J+n+4)(q+J-n+3)(q+J+n+3)}{(2q+5)G(q,n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
2	$-\frac{1}{2}$	1	$-\mathrm{i}\left[\frac{4J(J-1)(q+2)(q+J-n+3)(q+J+n+3)[(q+2)(q+5)-J(q+5)+n(q+3)]^2}{(q+4)(2q+5)G(q,n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
2	$-\frac{1}{2}$	2	$-\mathrm{i}\left[\frac{4(J-1)(J+1)(q+3)(q-J-n+2)(q-J+n+2)[(q+2)(q+3)+J(q+2)+n(q+5)]^2}{(q+4)(2q+5)G(q,n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
α'	k	α	$q+n+J=$ odd
1	$\frac{1}{2}$	1	0
2	$\frac{1}{2}$	1	$\mathrm{i}\left[\frac{G(q,n+1/2,J+1/2)}{(q+4)(2q+5)}\right]^{1/2}$
1	$-\frac{1}{2}$	1	$-\left[\frac{12(q+2)(q-J-n+3)(q+J+n+4)}{(2q+5)G(q,-n-1/2,J-1/2)}\right]^{1/2}$
2	$-\frac{1}{2}$	1	$\left[\frac{4(J-1)(J+1)(2q+7)^2(q-J+n+3)(q+J-n+2)}{(q+4)(2q+5)G(q,-n-1/2,J-1/2)}\right]^{1/2}$

[a] $F(q,n,J) = (q+2)^2(q+3) - (q+2)J(J+1) - (q+3)n^2$.
[b] $G(q,n,J) = (q+2)(4q+15) + (4q+17)n + [4q^2+30q+53+2(2q+7)(J+n)]J$.

and

$$\langle 1q'n_1'n_2'j_1'j_2'J' \,\|[\mathbf{R}\times\mathbf{S}^{+}]^{(1/2)}\|\, 2qn_1n_2j_1j_2J\rangle = \sqrt{2(2J'+1)(2J+1)}\begin{Bmatrix} j_1 & j_2 & J \\ \frac{1}{2} & 1 & \frac{1}{2} \\ j_1' & j_2' & J' \end{Bmatrix}$$
$$\times\langle (1/2,1/2)n_1'j_1'\|\mathbf{R}\|(1,1)n_1j_1\rangle\langle (q',0)n_2'j_2'\|\mathbf{S}^{+}\|(q,0)n_2j_2\rangle. \qquad (43)$$

The coefficients $A_{\alpha nJ}(n_1n_2j_1j_2)$ are shown in table A1 for representations $(q+1/2, 1/2)$ and in tables A2 and A3 for representations $(q+1, 1)$. The matrix elements on the right-hand side of (42) and (43) are given in appendix B. The final result is presented in tables 4 and 5.

5. Conclusions

In this article a new set of irreducible shift tensor operators under the non-canonical reduction of the compact symplectic group $Sp(4) \sim O(5) \supset SU(2) \times U(1)$ was defined, for the case when the tensor rank (ω_1, ω_2) corresponds to a non-degenerate representation. The tensors of rank $(1/2, 1/2)$, $(1, 1)$ and $(3/2, 3/2)$ were explicitly presented, tensors of other ranks of physical interest can be derived similarly. The construction was presented in the form of $SU(2)$ tensor products of certain elementary bosonic operators used previously by the author

Table 3. Matrix elements $\langle (q+3/2, 3/2)\alpha', n-1/2, J+k \| \mathbf{R}^{+} \| (q+1, 1)\alpha n J\rangle^{a}$.

α'	k	α	$q+n+J=$ even
1	$\frac{1}{2}$	1	$-\left[\frac{12(J+1)(J+2)(q-J-n+2)(q+J-n+5)[(q+2)(q+3)+J(q+2)+n(q+3)]^2}{(2q+5)G(q,n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
1	$\frac{1}{2}$	2	$\left[\frac{12J(J+2)(q+2)(q+3)(q-J+n+2)(q+J-n+3)(q+J-n+5)(q+J+n+3)}{(2q+5)G(q,n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
2	$\frac{1}{2}$	1	$\left[\frac{4J(J+1)(q+2)(q-J+n+2)(q+J+n+3)[q^2+8q+18+J(q+5)-n(q+3)]^2}{(q+4)(2q+5)G(q,n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
2	$\frac{1}{2}$	2	$\left[\frac{4(q+3)(q-J-n+2)(q+J-n+3)[3(q+2)(q+4)+J^2(q+2)+J(q^2+8q+12)+Jn(q+5)+3n(q+4)]^2}{(q+4)(2q+5)G(q,n-1/2,J+1/2)F(q,n,J)}\right]^{1/2}$
1	$-\frac{1}{2}$	1	$-\mathrm{i}\left[\frac{12J(J+1)(q-J-n+4)(q-J+n+2)[(q+2)(q+3)+J(q+2)-n(q+3)]^2}{(2q+5)G(q,-n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
1	$-\frac{1}{2}$	2	$-\mathrm{i}\left[\frac{12J^2(q+2)(q+3)(q-J-n+2)(q-J-n+4)(q+J-n+3)(q+J+n+3)}{(2q+5)G(q,-n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
2	$-\frac{1}{2}$	1	$\mathrm{i}\left[\frac{4J(J-1)(q+2)(q+J-n+3)(q+J+n+3)[(q+2)(q+5)-J(q+5)-n(q+3)]^2}{(q+4)(2q+5)G(q,-n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
2	$-\frac{1}{2}$	2	$-\mathrm{i}\left[\frac{4(J-1)(J+1)(q+3)(q-J-n+2)(q-J+n+2)[(q+2)(q+3)+J(q+2)-n(q+5)]^2}{(q+4)(2q+5)G(q,-n+1/2,J-1/2)F(q,n,J)}\right]^{1/2}$
α'	k	α	$q+n+J=$ odd
1	$\frac{1}{2}$	1	0
2	$\frac{1}{2}$	1	$\mathrm{i}\left[\frac{G(q,-n+1/2,J+1/2)}{(q+4)(2q+5)}\right]^{1/2}$
1	$-\frac{1}{2}$	1	$-\left[\frac{12(q+2)(q-J+n+3)(q+J-n+4)}{(2q+5)G(q,n-1/2,J-1/2)}\right]^{1/2}$
2	$-\frac{1}{2}$	1	$\left[\frac{4(J-1)(J+1)(2q+7)^2(q-J-n+3)(q+J+n+2)}{(q+4)(2q+5)G(q,n-1/2,J-1/2)}\right]^{1/2}$

[a] Functions $F(q, n, J)$ and $G(q, n, J)$ as in table 2.

Table 4. $\left\langle (q'+1/2, 1/2)n+1/2, J+k \left\| \mathbf{Q}a_{13} + \mathrm{i}\sqrt{\tfrac{3}{2}}[\mathbf{R}\times\mathbf{S}^{+}]^{(\frac{1}{2})} \right\| (q+1, 1)\alpha n J\right\rangle^{a}$.

k	α	$q+n+J=$ even
$\frac{1}{2}$	1	$-\mathrm{i}\left[\frac{2(J+1)(q+2)(q-J-n+2)[(q+2)(q+3)+J(q+2)+n(q+3)]^2}{(2q+5)F(q,n,J)}\right]^{1/2}$
$\frac{1}{2}$	2	$\mathrm{i}\left[\frac{2J(q+2)^2(q+3)(q-J+n+2)(q+J-n+3)(q+J+n+3)}{(2q+5)F(q,n,J)}\right]^{1/2}$
$-\frac{1}{2}$	1	$-\left[\frac{2J(q+2)(q+J-n+3)[(q+2)^2-J(q+2)+n(q+3)]^2}{(2q+5)F(q,n,J)}\right]^{1/2}$
$-\frac{1}{2}$	2	$-\left[\frac{2(J+1)(q+2)^2(q+3)(q-J-n+2)(q-J+n+2)(q+J+n+3)}{(2q+5)F(q,n,J)}\right]^{1/2}$
k	α	$q+n+J=$ odd
$\frac{1}{2}$	1	$-\left[\frac{2J(q+2)^2(q+J+n+4)}{2q+5}\right]^{1/2}$
$-\frac{1}{2}$	1	$-\mathrm{i}\left[\frac{2(J+1)(q+2)^2(q-J+n+3)}{(2q+5)}\right]^{1/2}$

[a] Function $F(q, n, J)$ as in table 2.

Table 5. $\left\langle (q'+1/2,1/2)n-1/2, J+k \left\| -\mathbf{R}a_{24} - \mathrm{i}\sqrt{\tfrac{3}{2}}[\mathbf{Q}\times\mathbf{S}^{+}]^{(\frac{1}{2})} \right\| (q+1,1)\alpha n J \right\rangle^{\mathrm{a}}$.

k	α	$q+n+J=$ even
$\frac{1}{2}$	1	$-\mathrm{i}\left[\frac{2(J+1)(q+2)(q-J+n+2)[(q+2)(q+3)+J(q+2)-n(q+3)]^2}{(2q+5)F(q,n,J)}\right]^{1/2}$
$\frac{1}{2}$	2	$-\mathrm{i}\left[\frac{2J(q+2)^2(q+3)(q-J-n+2)(q+J-n+3)(q+J+n+3)}{(2q+5)F(q,n,J)}\right]^{1/2}$
$-\frac{1}{2}$	1	$-\left[\frac{2J(q+2)(q+J+n+3)[(q+2)^2-J(q+2)-n(q+3)]^2}{(2q+5)F(q,n,J)}\right]^{1/2}$
$-\frac{1}{2}$	2	$\left[\frac{2(J+1)(q+2)^2(q+3)(q-J-n+2)(q-J+n+2)(q+J-n+3)}{(2q+5)F(q,n,J)}\right]^{1/2}$
k	α	$q+n+J=$ odd
$\frac{1}{2}$	1	$\left[\frac{2J(q+2)^2(q+J-n+4)}{2q+5}\right]^{1/2}$
$-\frac{1}{2}$	1	$\mathrm{i}\left[\frac{2(J+1)(q+2)^2(q-J-n+3)}{(2q+5)}\right]^{1/2}$

[a] Function $F(q,n,J)$ as in table 2.

in the derivation of the irreducible representations of $Sp(4) \supset SU(2)\times U(1)$. The components of the tensor operators have a well-defined rank under the $SU(2)$ subgroup of $Sp(4)$, so that the computation of the matrix elements can be performed in two steps. Initially, the reduced matrix elements can be derived in the representation space of the product $(p/2, p/2)\times(q,0)$, with the use of the composition formulae of the $SU(2)$ tensor algebra. Secondly, these reduced matrix elements can be projected onto the space of the arbitrary representation $(q+p/2, p/2) \subset (p/2, p/2)\times(q,0)$, using the set of coefficients $A_{\alpha nJ}(n_1n_2j_1j_2)$. This can be performed at least in the cases when the initial and final states of the matrix elements belong to representations with multiplicity equal to, or less than, 3. More general cases could be treated similarly, but the complexity of the computations noticeably increases.

The simplicity of the present method was demonstrated in the examples presented in section 4. The computation of the matrix elements of the generators given in [3] can be seen as another more complete example, due to their relation with the tensor $T^{(1,1)}_{000}(0,0)$, shown at the end of section 3.

Acknowledgments

The author wishes to thank Professor Jose Mestre and Professor Bill Gerace for the hospitality of the Scientific Reasoning Research Institute at the University of Massachusetts-Amherst where this work was completed.

Appendix A

In this appendix the coefficients $A_{\alpha nJ}(n_1n_2j_1j_2)$ for representations $(q+1/2, 1/2)$, $(q+1, 1)$ and $(q+3/2, 3/2)$ are presented. They were computed as solutions of the systems of algebraic equations derived in [3]. The case of representations $(q+2, 2)$ was obtained in a similar manner, but is not presented here for brevity. In general, the values of these coefficients are

Table A1. Coefficients $A_{nJ}(n_1 n_2 j_1 j_2)$ in $(q + 1/2, 1/2)$.

n_1	j_1	k	$A^o_{nJ}\left(n_1, \frac{n-n_1}{2}, j_1, J-k\right)$	$A^e_{nJ}\left(n_1, \frac{n-n_1}{2}, j_1, J+k\right)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\left(\frac{q+J+n+2}{2q+3}\right)^{1/2}$	$-\mathrm{i}\left(\frac{q-J+n+1}{2q+3}\right)^{1/2}$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\mathrm{i}\left(\frac{q-J-n+1}{2q+3}\right)^{1/2}$	$\left(\frac{q+J-n+2}{2q+3}\right)^{1/2}$

Table A2. Coefficients $A^o_{nJ}(n_1 n_2 j_1 j_2)$ in $(q + 1, 1)$.

n_1	j_1	k	$A^o_{nJ}\left(n_1, \frac{n-n_1}{2}, j_1, J+k\right)$
1	1	0	$-\mathrm{i}\left[\frac{(q-J+n+1)(q+J+n+2)}{2(q+2)(2q+3)}\right]^{1/2}$
0	1	1	$-\left[\frac{J(q-J-n+1)(q-J+n+1)}{(2J+1)(q+2)(2q+3)}\right]^{1/2}$
0	1	-1	$\left[\frac{(J+1)(q+J-n+2)(q+J+n+2)}{(2J+1)(q+2)(2q+3)}\right]^{1/2}$
-1	1	0	$-\mathrm{i}\left[\frac{(q-J-n+1)(q+J-n+2)}{2(q+2)(2q+3)}\right]^{1/2}$

Table A3. Coefficients $A^e_{\alpha nJ}(n_1 n_2 j_1 j_2)$ in $(q + 1, 1)$[a].

n_1	j_1	k	$A^e_{1nJ}\left(n_1, \frac{n-n_1}{2}, j_1, J+k\right)$
1	1	1	$-\mathrm{i}\left[\frac{(J+1)(q-J-n+2)(q-J+n)(q-J+n+2)(q+J+n+3)}{(4J+2)(2q+3)F(q,n,J)}\right]^{1/2}$
1	1	-1	$-\mathrm{i}\left[\frac{J(q-J+n+2)(q+J-n+3)(q+J+n+1)(q+J+n+3)}{(4J+2)(2q+3)F(q,n,J)}\right]^{1/2}$
0	1	0	0
0	0	0	$\left[\frac{(q-J-n+2)(q-J+n+2)(q+J-n+3)(q+J+n+3)}{(2q+3)F(q,n,J)}\right]^{1/2}$
-1	1	1	$\mathrm{i}\left[\frac{(J+1)(q-J-n)(q-J-n+2)(q-J+n+2)(q+J-n+3)}{(4J+2)(2q+3)F(q,n,J)}\right]^{1/2}$
-1	1	-1	$\mathrm{i}\left[\frac{J(q-J-n+2)(q+J-n+1)(q+J-n+3)(q+J+n+3)}{(4J+2)(2q+3)F(q,n,J)}\right]^{1/2}$
n_1	j_1	k	$A^e_{2nJ}\left(n_1, \frac{n-n_1}{2}, j_1, J+k\right)$
1	1	1	$\mathrm{i}\left[\frac{J(q-J+n)(q+J-n+3)[(q+2)^2-(q+2)J+(q+3)n]^2}{(4J+2)(q+2)(q+3)(2q+3)F(q,n,J)}\right]^{1/2}$
1	1	-1	$-\mathrm{i}\left[\frac{(J+1)(q-J-n+2)(q+J+n+1)[(q+2)(q+3)+(q+2)J+(q+3)n]^2}{(4J+2)(q+2)(q+3)(2q+3)F(q,n,J)}\right]^{1/2}$
0	1	0	$-\left[\frac{F(q,n,J)}{(q+2)(q+3)(2q+3)}\right]^{1/2}$
0	0	0	$-\left[\frac{J(J+1)n^2(2q+5)^2}{(q+2)(q+3)(2q+3)F(q,n,J)}\right]^{1/2}$
-1	1	1	$\mathrm{i}\left[\frac{J(q-J-n)(q+J+n+3)[(q+2)^2-(q+2)J-(q+3)n]^2}{(4J+2)(q+2)(q+3)(2q+3)F(q,n,J)}\right]^{1/2}$
-1	1	-1	$-\mathrm{i}\left[\frac{(J+1)(q-J+n+2)(q+J-n+1)[(q+2)(q+3)+(q+2)J-(q+3)n]^2}{(4J+2)(q+2)(q+3)(2q+3)F(q,n,J)}\right]^{1/2}$

[a] Function $F(q, n, J)$ as in table 2.

different depending on whether the sum $q + J + n$ is an odd or an even integer. They are distinguished by a superscript 'o' or 'e'.

Table A4. Coefficients $A_{\alpha n J}(n_1 n_2 j_1 j_2)$ in $(q+3/2, 3/2)$[a].

n_1	j_1	k	$A^e_{\alpha n J}\left(n_1, \frac{n-n_1}{2}, j_1, J+k\right)$	n_1	j_1	k	$A^o_{\alpha n J}\left(n_1, \frac{n-n_1}{2}, j_1, J+k\right)$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$v_{1\alpha}(q, n, J)$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$v_{9\alpha}(q, -n, J)$
$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$v_{2\alpha}(q, n, J)$	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$v_{10\alpha}(q, -n, J)$
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$v_{3\alpha}(q, n, J)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$v_{6\alpha}(q, -n, J)$
$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$v_{4\alpha}(q, n, J)$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$v_{7\alpha}(q, -n, J)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$v_{5\alpha}(q, n, J)$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-v_{8\alpha}(q, -n, J)$
$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$v_{6\alpha}(q, n, J)$	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$v_{3\alpha}(q, -n, J)$
$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$v_{7\alpha}(q, n, J)$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$v_{4\alpha}(q, -n, J)$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$v_{8\alpha}(q, n, J)$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-v_{5\alpha}(q, -n, J)$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$v_{8\alpha}(q, n, J)$	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$v_{1\alpha}(q, -n, J)$
$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$v_{10\alpha}(q, n, J)$	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$v_{2\alpha}(q, -n, J)$

[a] Functions $v_{i\alpha}$ are defined in table A5.

Table A5. Functions $v_{i\alpha}$ for representations $(q+3/2, 3/2)$, $\alpha = 1, 2$[a,b,c,d].

i	$v_{i\alpha}(q, n, J)$
1	$-\left[\frac{(J+1)[(q-J+n)^2-1]}{(2J+3)(q+J-n+4)}\right]^{\frac{1}{2}} s_\alpha(q, n, J) - \left[\frac{J(2J-1)(q-J-n+2)[(q-J+n)^2-1]}{3(2J+3)(q+J-n+4)(q+J+n+1)}\right]^{\frac{1}{2}} t_\alpha(q, n, J)$
2	$[J(q-J+n+1)]^{\frac{1}{2}} t_\alpha(q, n, J)$
3	$-\mathrm{i}\left[\frac{J(q-J+n+1)}{2J+3}\right]^{\frac{1}{2}} s_\alpha(q, n, J) - \mathrm{i}\left[\frac{3(J+1)(2J-1)(q-J-n+2)(q-J+n+1)}{(2J+3)(q+J+n+1)}\right]^{\frac{1}{2}} t_\alpha(q, n, J)$
4	$\mathrm{i}[(J+1)(q+J-n+2)]^{\frac{1}{2}} t_\alpha(q, n, J)$
5	$-\mathrm{i}(q-J+n+1)^{\frac{1}{2}} s_\alpha(q, n, J)$
6	$\left[\frac{3(q-J-n)(q-J+n+1)}{4(J+1)(2J+3)(q+J+n+3)}\right]^{\frac{1}{2}} s_\alpha(q, n, J) - \left[\frac{J(2J-1)(q-J-n)(q-J-n+2)(q-J+n+1)}{(2J+3)(q+J+n+1)(q+J+n+3)}\right]^{\frac{1}{2}} t_\alpha(q, n, J)$
7	$\left[\frac{(2J-1)(q+J-n+2)}{4(J+1)}\right]^{\frac{1}{2}} s_\alpha(q, n, J) + \left[\frac{3J(q+J-n+2)(q-J-n+2)}{q+J+n+1}\right]^{\frac{1}{2}} t_\alpha(q, n, J)$
8	$(q+J-n+2)^{\frac{1}{2}} s_\alpha(q, n, J)$
9	$\mathrm{i}\left[\frac{3(q-J-n)(q+J-n+2)}{4J(2J+3)(q+J+n+3)}\right]^{\frac{1}{2}} s_\alpha(q, n, J) - \mathrm{i}\left[\frac{(J+1)(2J-1)(q-J-n)[(q-n+2)^2-J^2]}{(2J+3)(q+J+n+1)(q+J+n+3)}\right]^{\frac{1}{2}} t_\alpha(q, n, J)$
10	$\mathrm{i}\left[\frac{(2J-1)(q+J-n)(q+J-n+2)}{4J(q-J+n+3)}\right]^{\frac{1}{2}} s_\alpha(q, n, J) + \mathrm{i}\left[\frac{(J+1)(q+J-n)[(q-n+2)^2-J^2]}{3(q-J+n+3)(q+J+n+1)}\right]^{\frac{1}{2}} t_\alpha(q, n, J)$

[a] $s_1(q, n, J) = \left[\frac{2(2J+3)(q-J+n+3)(q+J-n+4)(q+J+n+3)}{(2q+3)(2q+5)G(q,n,J)}\right]^{\frac{1}{2}}$, $t_1(q, n, J) = 0$.

[b] $s_2(q, n, J) = -\left[\frac{(2J-1)(q-J-n+2)[z(q,n,J)]^2}{6(q+2)(q+4)(2q+3)(2q+5)G(q,n,J)}\right]^{\frac{1}{2}}$, $t_2(q, n, J) = \left[\frac{(q+J+n+1)G(q,n,J)}{8J(J+1)(q+2)(q+4)(2q+3)(2q+5)}\right]^{\frac{1}{2}}$.

[c] $z(q, n, J) = 2q^2 + 16q + 33 + 2J(q+5) + 2n(q+2)$.

[d] Function $G(q, n, J)$ as in table 2.

The basis states of $(q+1/2, 1/2)$ are not degenerate, so that the parameter α is irrelevant. Table A1 shows the corresponding coefficients.

Basis states of $(q+1, 1)$ with $q+J+n = odd$ are also not degenerate. Those with $q+J+n = even$ have, in general, multiplicity 2. Their $A_{\alpha n J}$ coefficients are given in tables A2 and A3.

In representations $(q+3/2, 3/2)$ the basis vectors have, in general, multiplicity 2. Their coefficients $A_{\alpha n J}$ are given in table A4 through the functions $v_{i\alpha}$ of table A5.

Appendix B

The operators $\mathbf{Q}^{(1/2)}$, $\mathbf{R}^{(1/2)}$, $\mathbf{Q}^{(1/2)+}$ and $\mathbf{R}^{(1/2)+}$ act on the representation space of $\left(\frac{p}{2}, \frac{p}{2}\right)$. Their non-zero reduced matrix elements were given in [3]. For the sake of completeness, and due to a change in the phase factors, they are given again in this appendix.

$$\left\langle\left(\frac{p-1}{2}, \frac{p-1}{2}\right) n - \frac{1}{2}, j + \frac{1}{2}\right\| \mathbf{Q} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = -\mathrm{i}\left[\left(\frac{p}{2} - j\right)(j - n + 1)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p-1}{2}, \frac{p-1}{2}\right) n - \frac{1}{2}, j - \frac{1}{2}\right\| \mathbf{Q} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = -\mathrm{i}\left[\left(\frac{p}{2} + j + 1\right)(j + n)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p-1}{2}, \frac{p-1}{2}\right) n + \frac{1}{2}, j + \frac{1}{2}\right\| \mathbf{R} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = \mathrm{i}\left[\left(\frac{p}{2} - j\right)(j + n + 1)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p-1}{2}, \frac{p-1}{2}\right) n + \frac{1}{2}, j - \frac{1}{2}\right\| \mathbf{R} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = -\mathrm{i}\left[\left(\frac{p}{2} + j + 1\right)(j - n)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p+1}{2}, \frac{p+1}{2}\right) n + \frac{1}{2}, j + \frac{1}{2}\right\| \mathbf{Q}^{+} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = -\left[\left(\frac{p}{2} + j + 2\right)(j + n + 1)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p+1}{2}, \frac{p+1}{2}\right) n + \frac{1}{2}, j - \frac{1}{2}\right\| \mathbf{Q}^{+} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = \left[\left(\frac{p}{2} - j + 1\right)(j - n)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p+1}{2}, \frac{p+1}{2}\right) n - \frac{1}{2}, j + \frac{1}{2}\right\| \mathbf{R}^{+} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = -\left[\left(\frac{p}{2} + j + 2\right)(j - n + 1)\right]^{\frac{1}{2}}$$

$$\left\langle\left(\frac{p+1}{2}, \frac{p+1}{2}\right) n - \frac{1}{2}, j - \frac{1}{2}\right\| \mathbf{R}^{+} \left\| \left(\frac{p}{2}, \frac{p}{2}\right) nj\right\rangle = -\left[\left(\frac{p}{2} - j + 1\right)(j + n)\right]^{\frac{1}{2}}.$$

For the same reasons the non-zero reduced matrix elements of the $SU(2)$ vector $\mathbf{S}^{(1)}$ and $SU(2)$ scalars $\bar{a}_{13}$, and $\bar{a}_{24}$ are shown here. Their action is restricted to the representation space $(q, 0)$.

$$\langle (q-1, 0)n, j+1 \| \mathbf{S} \| (q, 0)nj\rangle = -\left[\frac{(j+1)(q+1)(q-j+2n)(q-j-2n)}{2q+1}\right]^{\frac{1}{2}}$$

$$\langle (q-1, 0)n, j-1 \| \mathbf{S} \| (q, 0)nj\rangle = -\left[\frac{j(q+1)(q+j+2n+1)(q+j-2n+1)}{2q+1}\right]^{\frac{1}{2}}$$

$$\left\langle (q-1, 0)n - \tfrac{1}{2}, j \left\| \bar{a}_{13} \right\| (q, 0)nj\right\rangle = \left[\frac{(2j+1)(q+1)(q+j+2n+1)(q-j+2n)}{2(2q+1)}\right]^{\frac{1}{2}}$$

$$\left\langle (q-1, 0)n + \tfrac{1}{2}, j \left\| \bar{a}_{24} \right\| (q, 0)nj\right\rangle = \left[\frac{(2j+1)(q+1)(q+j-2n+1)(q-j-2n)}{2(2q+1)}\right]^{\frac{1}{2}}.$$

The reduced matrix elements of the Hermitian-conjugated $\mathbf{S}^{(1)+}$, a_{13} and a_{24} can be computed immediately with the usual expression for tensors of integer rank:

$$\langle (q', 0)n'j' \| \mathbf{T}^{(k)} \| (q, 0)nj\rangle^{*} = (-1)^{j'-j} \langle (q, 0)nj \| \mathbf{T}^{(k)+} \| (q', 0)n'j'\rangle.$$

References

[1] Bincer A M 1980 *J. Math. Phys.* **21** 671

[2] Hecht K T 1967 *Nucl. Phys.* A **102** 11

[3] Tello-Llanos R A 2002 *J. Phys. A: Math. Gen.* **35** 1079

[4] Holman W J 1969 *J. Math. Phys.* **10** 1710

[5] Hecht K T 1993 *J. Phys. A: Math. Gen.* **26** 329